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On the local solvability of elliptic equations on compact manifolds

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Abstract

The local image of a nonlinear elliptic operator on a compact manifold is a submanifold described by a full set of independent equations if and only if the corank of the linearized operator is constant. When not so, we exhibit a higher order infinitesimal invariant, the epidimension, which forces the number of independent equations decrease. We show that the epidimension of a natural operator with enough symmetry must either vanish or be maximal, in which case the local image admits no equation. In general, we show that a local nonlinear version of Fredholm's scheme, which always exists, encodes the maximal number of independent equations. Finally, we take a glimpse at the underdetermined elliptic case and state a conjecture for it.

Résumé

L'image locale d'un opérateur elliptique non linéaire sur une variété compacte est une sous-variété décrite par un ensemble complet d'équations indépendantes si et seulement si le corang de l'opérateur linéarisé est constant. Quand ce n'est pas le cas, on exhibe un invariant infinitésimal d'ordre supérieur, l'épidimension, qui force le nombre d'équations indépendantes à décroitre. On montre que l'épidimension d'un opérateur naturel avec assez de symétrie doit, ou bien s'annuler, ou être maximale, auquel cas l'image locale n'admet aucune équation. En général, on montre qu'une version non linéaire locale du schéma de Fredholm, qui existe toujours, fournit le nombre maximal d'équations indépendantes. Enfin, on décrit brièvement le cas sous-déterminé elliptique, pour lequel une conjecture est énoncée.

MSC: 58J05; 58K05

Keywords: Nonlinear elliptic operators; Codimension; Local image; Maximal constraints; Local Fredholm resolution

1. Introduction

We continue here the semi-abstract study initiated in [8, Sections 2–3] and bring sharp results. To recall our purpose, let F be a nonlinear differential operator on a compact connected manifold M (without boundary), acting between Fréchet spaces $\mathcal{E}$ (source) and $\mathcal{F}$ (target) of sections of vector bundles on M (all objects are smooth). We assume that the linearization $L = dF(u_0)$ of F at $u_0 \in \mathcal{E}$ is elliptic with a nonzero cokernel of dimension c_0 and we take $u_0 = 0$, $F(u_0) = 0$ unless otherwise specified. In that situation, we are interested in the following questions: for which

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sections $f \in \mathcal{F}$ close to 0 can one solve the equation $F(u) = f$ with u close to 0 in $\mathcal{E}$? Under which condition on F can one characterize the so-called local image of F at 0, namely $\mathcal{I} = \{F(u),\ u \text{ near } 0 \text{ in } \mathcal{E}\}$, by c_0 independent equations (what we call a full constraint)?

Note that we restrict to one operator for simplicity (for standard results on elliptic operators, see e.g. [3, Appendix] and references therein), but could deal with nonlinear strongly elliptic systems as well [16, p. 671].

These questions are nothing but an abstract form of the simplest ones arising in differential geometry, each time one is trying to solve a nontrivial inverse problem, for instance the k-Minkowski problem [12], which involves on the standard n-sphere the scalar operator $F(u) = \sigma_k[\kappa(u)]$, where $\kappa(u) = (\frac{1}{R_1}, \dots, \frac{1}{R_n})$ with the R_α's solving the characteristic equation:

$$\det\bigl[\operatorname{Hess} u + (1 + u - R_\alpha)I\bigr] = 0.$$

Here, “Hess” stands for the Hessian (endomorphism) operator and σ_k, for the elementary symmetric function of order k. The kernel of $v \mapsto \operatorname{Hess} v + vI$ on the standard sphere is well known, denoted by Λ_1, generated by the restriction to $\mathbb{S}^n$ of the ambiant coordinate functions x_i of $\mathbb{R}^{n+1}$ (first spherical harmonics, see e.g. [2]). One may thus restrict F to $\Lambda_1^\perp$ with no loss of generality. Calculation shows that $L = dF(0)$ is proportional to the trace of $(\operatorname{Hess} + I)$, hence self-adjoint with kernel equal to Λ_1: so $c_0 = n + 1$. If $k = n$ (genuine smooth Minkowski problem [4]), geometry yields for the local image $\mathcal{I}$ the required full constraint, namely (using on $\mathbb{S}^n$ the standard Lebesgue measure $d\omega$):

$$\forall i \in \{1, \dots, n\}, \quad \int_{\mathbb{S}^n} \frac{x_i}{F(u)(x)}\, d\omega \equiv 0.$$

For $k < n$, though, no constraint is known; symmetry assumptions had to be made in [12] in order to overcome this difficulty and solve on $\mathbb{S}^n$ the k-curvature equation $F(u) = f$. This example shows that our present study may be viewed as a preliminary step for attacking many geometric inverse problems.

Back to our general abstract setting, a full constraint is known to exist near 0 if the kernel of $dF(u)$ does not depend on u [8, Theorem 3], as it is the case for the preceding example [8, Proposition 8]. In Section 2, relying on Frobenius scheme to cope with non-flat situations, we improve that criterion and get the following optimal one: F admits a full constraint if and only if $dF(u)$ has constant corank.

When the corank of $dF[u]$ varies near $u_0 = 0$, all one can do is pick as many independent equations identically satisfied by $F[u]$ as possible, what we call a maximal constraint. The number of such equations, denoted by $\operatorname{cod}(F, 0)$, is called the local codimension of F at 0. In Section 3, we exhibit an infinitesimal condition on F at 0, the epijets criterion, which implies $\operatorname{cod}(F, 0) < c_0$ and discuss it with an example. The idea for that condition may be expressed as follows: if some higher order derivatives of F at 0 in the direction of $\ker L$ have nonzero projection (called epijets) onto the coimage of L, they kill part of the infinitesimal codimension c_0 of $\mathcal{I}$ at 0.

In Section 4, we illustrate the usefulness of the epijets criterion by the case of natural operators with enough symmetry: they admit either no epijets or c_0 independent ones (which precludes the existence of any good equation), a dichotomy encountered in landmark geometric problems (Nirenberg operator versus Minkowski type operators (as above)) [8, Sections 4.1–4.2].

But how can one construct a maximal constraint locally? A method, proposed by the author in case a full constraint exists, goes as follows. First, regardless of constraints, it is always possible to write down for F near 0 a nonlinear version of the Fredholm alternative equation, what we call a local Fredholm resolution [8], which reads (using auxiliary L^2 scalar products on $\mathcal{E}$ and $\mathcal{F}$ and setting L^* for the formal adjoint of L) like: for each f near 0 in $\mathcal{F}$, there exists an approximate inverse $f \mapsto S(f)$ ranging near 0 in $\mathcal{E}$ and a defect map $f \mapsto D(f) \in \ker L^*$ such that the equation

$$F\bigl[S(f)\bigr] = f - D(f)$$

identically holds. This result is established in [8, Theorem 6] by combining the linear Fredholm alternative theorem [3, Appendix] with the (C^∞ elliptic) inverse function theorem [5]. Now, if a full constraint exists, the defect map $f \mapsto D(f)$ must be another one [8, Theorem 2].

In Section 5, we extend the latter result to maximal constraints which are not full, by a suitable extraction process from the above defect map $f \mapsto D(f)$. In particular, a good approximation of a maximal constraint arises just by (linearly) projecting D onto an appropriate $\operatorname{cod}(F, 0)$-dimensional subspace of $\ker L^*$.

Finally, the Appendix is devoted to the case where F is an underdetermined elliptic operator. When so, all one can assert is that $dF(u)$ is semi-Fredholm, with finite corank; moreover, the constant corank condition is necessary for a full constraint to exist. We conjecture that it is sufficient. Besides, all one can show is the full constrainability of the restriction of F to the orthogonal of $\ker L$ (analogue of the second part (strongness) of [8, Theorem 6] in the determined case). We conclude with examples (underdetermined), one admitting a full constraint, the other none.

2. Full constraints

Before we proceed to state and prove the results announced, let us specify a few notations and definitions (mostly from [8, Section 2]). Henceforth, we let $\mathcal{V}$ be a small enough nonempty neighborhood of 0 in $\mathcal{F}$ and $\mathcal{U}$, the connected component of $F^{-1}(\mathcal{V}) \subset \mathcal{E}$ containing 0; we consider the local image $F(\mathcal{U})$, sometimes denoted by $\mathcal{I}(F,0)$ when $\mathcal{V}$ is unspecified arbitrarily small.

For $u \in \mathcal{U}$, the linear operator $dF(u)$ remains elliptic of constant index $i_u \equiv i_0$ [13, p. 235] and we denote by n_u (resp. c_u) the dimension of $\ker dF(u)$ (resp. $\operatorname{coker} dF(u)$), thus with $n_u - c_u = i_0$. Let us call c_u the *infinitesimal codimension* of F at u.

For later use, let us fix auxiliary L^2 scalar products on $\mathcal{E}$ and $\mathcal{F}$, set L^* for the formal adjoint of L, P_* for the orthogonal projection of $\mathcal{F}$ onto $\ker L^*$.

Definition 1. Given an integer $k \in \{1, \ldots, c_0\}$ and a k-dimensional real vector space $\mathbb{V}_k$, a *k-constraint* for F on $\mathcal{V}$ is a submersion $K : \mathcal{V} \to \mathbb{V}_k$ such that the identity: $K \circ F = 0$ is satisfied on $\mathcal{U}$. Let K be a k-constraint for F on $\mathcal{V}$; it is called *full* if $k = c_0$.

Remark 1. To see why $k \leqslant c_0$ in that definition, just linearize the identity $K \circ F = 0$ at any fixed $u \in \mathcal{U}$ and get k independent equations for the image of $dF(u)$, which is c_u-codimensional: we infer $k \leqslant c_u$; in particular $k \leqslant c_0$.

If $\ker dF(u)$ is constant on $\mathcal{U}$, there exists a full constraint for F on $\mathcal{V}$ [8, Theorem 3]. This result suffices to treat many geometric scalar second order operators (e.g. Calabi–Yau [1, p. 143], [19,6], Hessian [7], Minkowski [4], [8, p. 39]). In general, it should be improved though; our first result is an optimal improvement of it, namely:

Theorem 1. *There exists a full constraint for F on $\mathcal{V}$ if and only if the infinitesimal codimension of F remains constant on $\mathcal{U}$.*

Proof. The "only if" part is standard, due to the stability inequality $c_u \leqslant c_0$ [13, p. 235] combined with the inequality noted above $k \leqslant c_u$ for a k-constraint (here with $k = c_0$).

To prove the "if" part, we first infer the constancy condition $n_u \equiv n_0$ and note that the n_0-dimensional distribution $u \mapsto \ker dF(u)$ is stable on $\mathcal{U}$ under the vector fields bracket, as a straightforward consequence of the symmetry of the second differential of F. Frobenius theorem [15], applied in appropriate Banach completions $\bar{\mathcal{E}}$ and $\bar{\mathcal{F}}$ of $\mathcal{E}$ and $\mathcal{F}$ (e.g. using Hölder norms of sufficiently high orders, depending upon the order of the operator F, and such that our neighborhoods $\mathcal{U}$ and $\mathcal{V}$ are traces, respectively on $\mathcal{E}$ and $\mathcal{F}$, of neighborhoods $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ in $\bar{\mathcal{E}}$ and $\bar{\mathcal{F}}$), combined with nonlinear elliptic regularity theory (according to which, if $u \in \widetilde{\mathcal{U}}$ satisfies $F(u) = v \in \mathcal{V}$, then $u \in \mathcal{U}$, see [11, p. 532] or [3, Theorem 41]), Frobenius theorem thus, implies that $\mathcal{U}$ is foliated by n_0-dimensional integral submanifolds of that distribution, with each leaf sent by F to a single point of $\mathcal{V}$.

Using our auxiliary L^2 scalar products and setting $\bar{F}$ for the restriction of F to $\mathcal{U} \cap (\ker L)^{\perp}$, we may assume that $(\ker L)^{\perp}$ is transversal to the foliation, therefore $F(\mathcal{U})$ coincides with the image of $\bar{F}$. Furthermore, Fredholm theorem [3, Appendix] shows that the linear map $d\bar{F}(0) : (\ker L)^{\perp} \to \mathcal{F}$ is direct one-to-one with (closed) c_0-codimensional image, equal to $(\ker L^*)^{\perp}$. A classical argument [15, Section I.5], used here with the C^∞ elliptic inverse function theorem [5], provides a diffeomorphism $\Psi : \mathcal{V} \to (\ker L)^{\perp} \oplus \ker L^*$ such that the composed map $\Psi \circ \bar{F}$ reads like the canonical embedding $(\ker L)^{\perp} \to (\ker L)^{\perp} \oplus \ker L^*$. Composing Ψ with the canonical projection $(\ker L)^{\perp} \oplus \ker L^* \to \ker L^*$, we thus obtain a map $K : \mathcal{V} \to \ker L^*$ which satisfies $K \circ F = 0$ on $\mathcal{U}$. $\square$

Finally, let us emphasize that the criterion of Theorem 1 may be written as well: $n_u \equiv n_0$, due to the constancy of the index, but also $c_0 = \inf_{u \in \mathcal{U}} c_u$ (then $c_u \equiv c_0$ by the stability inequality $c_u \leqslant c_0$) [13, p. 235].

Henceforth, we focus on the situation where the criterion of Theorem 1 is *not* fulfilled: c_u varies with $u \in \mathcal{U}$, there exists no full constraint for F on $\mathcal{V}$.

3. Maximal constraints: the epijets criterion

Given $F:\mathcal{U}\to\mathcal{V}$ as above, the largest integer k such that there exists a k-constraint for F on $\mathcal{V}$ is called *the codimension of* F on $\mathcal{V}$, denoted by $\operatorname{cod}(F,\mathcal{V})$. Obviously, as $\mathcal{V}$ decreases (for the neighborhoods of 0 inclusion order), $\operatorname{cod}(F,\mathcal{V})$ is non-decreasing and bounded above by c_0 (cf. Remark 1). Setting $(\mathcal{V}_i)_{i\in\mathbb{N}}$ for a decreasing sequence of neighborhoods of 0, shrinking to 0 as $i\to\infty$, we infer the existence of an integer equal to $\lim_{i\to\infty}\operatorname{cod}(F,\mathcal{V}_i)$ which we call the *local codimension of* F *at* 0 (as opposed to the infinitesimal one c_0) and denote by $\operatorname{cod}(F,0)$. One readily checks that it does not depend on the choice of the sequence $(\mathcal{V}_i)_{i\in\mathbb{N}}$ and that it identically satisfies: $\operatorname{cod}(F,0)\geqslant\operatorname{cod}(F,\mathcal{V})$. From its very definition $\operatorname{cod}(F,0)$ represent the maximal number of independent equations for $\mathcal{I}(F,0)$; it is equal to $\operatorname{cod}(F,\mathcal{V})$ provided $\mathcal{V}$ is taken small enough.

Definition 2. A maximal constraint for F at 0 is a k-constraint for F on some neighborhood of 0 with $k=\operatorname{cod}(F,0)$.

We look for an infinitesimal condition on F at 0 implying $\operatorname{cod}(F,0)<c_0$. Let K be a maximal constraint for F at 0. As in [8, p. 27] we observe that the linearization at 0 of the constraint equation $K\circ F=0$ yields,

$$(\ker L^*)^\perp\subset\ker dK(0),$$

due to Fredholm theorem [3, Appendix]. It implies the identity:

$$\forall f\in\mathcal{V},\quad dK(0)(f)=dK(0)(P_*f),\tag{1}$$

and leads us to consider the $\operatorname{cod}(F,0)$ codimensional subspace $\perp^*(F,0)$ of $\ker L^*$ defined by:

$$\perp^*(F,0)=\bigl\{f\in\ker L^*,\ dK(0)(f)=0\bigr\}.$$

Clearly, the space $\perp^*(F,0)$ is independent of the particular maximal constraint chosen; we call it the epitangent space to F at 0. Now, the equality,

$$\operatorname{cod}(F,0)+\dim\perp^*(F,0)=c_0,$$

prompts us to exhibit nonzero elements of $\perp^*(F,0)$.

We require further notations. Given $u\in\mathcal{E}$, say with u a unit vector (for our L^2 product), set $F(tu)=\sum_{k=1}^\infty\frac{t^k}{k!}f_k(u)$ for the formal power expansion of $F(tu)$ at $t=0$ and note the identities: $\forall\lambda\in\mathbb{R}^*$, $\forall k\in\mathbb{N}$, $f_k(\lambda u)=\lambda^k f_k(u)$. In case $u=\xi\in\ker L$, the expansion of $F(t\xi)$ starts from $k=2$; we say that F is *flat* at 0 in the direction of ξ (or ξ-flat, for short), if all the $f_k(\xi)$'s vanish. If the manifold M and the vector bundles associated to $\mathcal{E}$ and $\mathcal{F}$ are real analytic, and if the operator F factors through a real analytic jet bundles map, then F cannot be ξ-flat for each $\xi\in\ker L$ unless it admits a full constraint near 0: indeed then, recalling the stability inequality $n_u\leqslant n_0$ [13, p. 235], $\ker dF(u)$ must be constant for u close to 0 in $\mathcal{E}$, which implies full constrainability [8, Theorem 3].

When F is not ξ-flat, we set $j_F(\xi)=j\geqslant 1$ for the first integer such that $f_j(\xi)=0$ but $f_{j+1}(\xi)\neq 0$ and call $j_F(\xi)$ the *order* of (the direction defined by) ξ relative to the operator F at 0. Recalling (1), we are lead to state:

Definition 3. A hypo-critical vector for F at 0 is a unit vector $\xi\in\ker L$ of finite order $j_F(\xi)$ such that

$$\exists k\in\bigl\{j_F(\xi)+1,\dots,2j_F(\xi)+1\bigr\},\quad P_*f_k(\xi)\neq 0;$$

such a nonzero vector $P_*f_k(\xi)$ is called an epijet of F at 0. The set of epijets for all hypo-critical vectors spans a subspace of $\ker L^*$ whose dimension is called the epidimension of F at 0, denoted by $\overline{\dim}(F,0)$.

Theorem 2. *The following inequality holds*:

$$\operatorname{cod}(F,0)\leqslant c_0-\overline{\dim}(F,0).\tag{2}$$

Proof. We will prove that each epijet of F at 0 lies in $\perp^*(F,0)$. Granted this fact, we get the inequality:

$$\dim\perp^*(F,0)\geqslant\overline{\dim}(F,0)$$

and the theorem follows. To complete the proof, fix a maximal constraint K for F at 0 and a hypo-critical vector ξ and set $j = j_F(\xi)$. Since $K \circ F = 0$, applying Lemma 1 below (Appendix A) with $G = F$ and $H = K$, we find:

$$\forall k \in \{j+1, \dots, 2j+1\}, \quad dK(0)\big[f_k(\xi)\big] = 0.$$

Combining the latter with (1), we infer:

$$\forall k \in \{j+1, \dots, 2j+1\}, \quad P_* f_k(\xi) \in \perp^* (F, 0),$$

and we are done. □

Remark 2. Inequality (2) for F at $u_0 = 0$ is stable under smooth perturbations (of F and u_0). Indeed, from its definition, $\overline{\dim}(F, 0)$ would not decrease when perturbed, while c_0 would [13, p. 235] thus forcing $\mathrm{cod}(F, 0)$ behave similarly.

Remark 3. One may wonder whether or not the epitangent space is spanned by the epijets. Let us provide an example showing that either situation may occur. Take $M = \mathbb{S}^2$ equipped with its standard metric g_0, and $F(u) = \Delta_0 u - 2u - u^p$, where u is a smooth real function on $\mathbb{S}^2$ close to 0, $p \geqslant 2$ an integer and Δ_0, the positive Laplacian of g_0. The linearized operator $L = dF(0) = \Delta_0 - 2$ is formally self-adjoint (for the L^2 structure relative to g_0) with a 3-dimensional kernel equal to Λ_1, the space of first spherical harmonics [2]; so $i_0 = 0$, $c_0 = 3$. Calculations yields:

$$\forall (u, \varphi) \in C^\infty(\mathbb{S}^2) \times C^\infty(\mathbb{S}^2), \quad dF(u)(\varphi) = \Delta_0 \varphi - 2\left(1 + \frac{p}{2} u^{p-1}\right)\varphi,$$

therefore, at $u = \varepsilon \neq 0$ constant close to 0, we get $\ker dF(\varepsilon) = \{0\}$, hence $c_\varepsilon = 0$. Recalling Remark 1, the latter implies $\mathrm{cod}(F, 0) = 0$; so $\dim \perp^* (F, 0)$ is maximal, equal to 3.

What about $\overline{\dim}(F, 0)$? One readily finds it (exercise) equal to 3 if p is odd, but vanishing if p is even. So, for p odd, the span of the epijets coincides with $\perp^* (F, 0)$, while for p even, it does not.

4. Dichotomy for natural operators with enough symmetry

In this section, we drop the normalization[2] $u_0 = 0$; besides, we stick to previous notations, thus with $L = dF(u_0)$. Furthermore, we assume the existence of a natural nonlinear differential operator $\mathbf{F}$, with identical source and target functor, and of a Riemannian metric g on M such that $F = \mathbf{F}(M, g)$ (apart from nonlinearity, the definition of such an operator $\mathbf{F}$ is that of [18, p. 656]). The naturality assumption requires that the Fréchet space $\mathcal{E}$ now consists of sections of a *tensor* bundle E over M.

Let G be a subgroup of the isometry group of (M, g). The metric g, its L^2 scalar product on M (denoted by $\langle .,. \rangle$), and the isometric action of G (both for g and for $\langle .,. \rangle$) all extend canonically to E [14]. In this context, we can prove the following dichotomy result:

Theorem 3. *If $u_0 \in \mathcal{E}$ is G-invariant and G acts irreducibly on* $\ker L \cap \ker L^*$, *then the epidimension of F at u_0 equals either* 0 *or* c_0 (*its largest possible value*); *in the latter case, F admits no constraint near u_0.*

That dichotomy is illustrated in [8] by two opposite types of curvature increment (scalar) operators F on the sphere, near the standard metric. For both types, $\ker L$ is the space Λ_1 of first spherical harmonics. On the one hand, if F is the increment of a Weingarten curvature operator considered in the *Minkowski* parametrization, then F is *flat* at 0 in any direction of Λ_1 (since it factors through the Codazzi operator, see [8, p. 38]); so it has no hypo-critical direction. On the other hand, if F is the conformal Gauss or scalar curvature increment operator (Nirenberg operator), hypo-critical directions do exist [8, p. 36].

Remark 4. If G acts irreducibly on $\ker L^*$, then $\ker L$ and $\ker L^*$ must either have zero intersection or coincide, because each is globally stable under the action of G, which we now show for completeness. Let us first prove the G-stability of $\ker L$. For each $(\varphi, v) \in G \times \mathcal{E}$ and for $t \in \mathbb{R}$ small, we have (setting $v \mapsto \varphi_* v$ for the G-action on $\mathcal{E}$):

$$\varphi_* F(u_0 + tv) = F(u_0 + t\varphi_* v), \tag{3}$$

[2] Because we will have to assume some invariance for u_0 always satisfied by 0.

by the naturality of F combined with the invariance of u_0. Differentiating both sides at $t = 0$, yields the commutation property:

$$\varphi_* L v \equiv L \varphi_* v. \tag{4}$$

In particular, if $v \in \ker L$, we get indeed $\varphi_* v \in \ker L$. Moreover, (4) combined with Fredholm theorem [3, Appendix] implies that $\ker L^{*\perp}$ is stable under the action of G, hence so is $\ker L^*$ recalling the (isometry) identity:

$$\forall (v, w) \in \mathcal{E}^2, \quad \langle \varphi_* v, \varphi_* w \rangle \equiv \langle v, w \rangle.$$

Finally, let us note the stability of $\ker L \cap \ker L^*$ under the action of G, in connection with condition (iii) of Theorem 4 below.

Proof of Theorem 3. Setting formally $F(u_0 + tv) = \sum_{k=1}^{\infty} \frac{t^k}{k!} f_k(v)$, we infer from (3) the identity:

$$\forall \varphi \in G,\ \forall v \in \mathcal{E}, \quad \varphi_* f_k(v) = f_k(\varphi_* v). \tag{5}$$

If $\overline{\dim}(F, u_0) \neq 0$, there exists $\xi \in \ker L$ hypo-critical; from (5), the order is constant along its orbit: $\forall \varphi \in G$, $j_F(\varphi_* \xi) \equiv j_F(\xi)$. Let $k \in \{j_F(\xi)+1, \dots, 2j_F(\xi)+1\}$ be such that $P_* f_k(\xi) \neq 0$. The identity (5) yields the equation:

$$\forall \varphi \in G, \quad P_* f_k(\varphi_* \xi) = P_* \varphi_* f_k(\xi),$$

the right-hand side of which is equal to $\varphi_*[P_* f_k(\xi)]$. Now the irreducibility assumption implies that the epijets subset,

$$\left\{\varphi_*\left[P_* f_k(\xi)\right],\ \varphi \in G\right\} \subset \ker L^*,$$

spans all of $\ker L^*$; in other words, $\overline{\dim}(F, u_0) = c_0$ as claimed. Furthermore, Theorem 2 implies $\operatorname{cod}(F, u_0) = 0$. □

In case $\ker L \cap \ker L^* \neq \{0\}$ with $\ker L \neq \ker L^*$, we can still prove the following:

Theorem 4. *Assume $u_0 \in \mathcal{E}$ is G-invariant and the following conditions hold*:

(i) $\ker L \cap \ker L^* \neq \{0\}$;
(ii) *there exists $\xi \in \ker L \cap \ker L^*$ such that F is not ξ-flat at u_0 and*

$$\exists k \in \left\{j_F(\xi)+1, \dots, 2j_F(\xi)+1\right\}, \quad \left\langle \xi, f_k(\xi)\right\rangle \neq 0;$$

(iii) *G acts irreducibly on* $\ker L \cap \ker L^*$.

Then we have the inequality:

$$\operatorname{cod}(F, u_0) \leqslant c_0 - \dim(\ker L \cap \ker L^*).$$

Typical examples fulfilling the assumptions of Theorem 4 (with $j = 1$) on the standard sphere are: the Nirenberg (conformal Gauss curvature) operator [8, Section 4] and its fully nonlinear analogues [9], Q-curvature operators (of any degree) [10].

Remark 5. Condition (ii) of Theorem 4 typically holds, possibly under some curvature assumptions on (M, g), when k is odd and f_k coercive, namely when:

$$\exists \theta > 0,\ \forall v \in \mathcal{E}, \quad \left\langle v, f_k(v)\right\rangle \geqslant \theta |v|_{L^2}^2.$$

This is the case, with $k = 3$, for the examples just quoted [8–10].

Proof of Theorem 4. From (5) written with $v = \xi$ satisfying assumption (ii), we infer:

$$\forall \varphi \in G, \quad \left\langle \varphi_* \xi, f_k(\varphi_* \xi)\right\rangle = \left\langle \xi, f_k(\xi)\right\rangle,$$

showing that $\varphi_* \xi$ also satisfies condition (ii). Since the orbit $\{\varphi_* \xi,\ \varphi \in G\}$ spans the whole of $\ker L \cap \ker L^*$, due to condition (iii), we readily infer from condition (ii) the inequality:

$$\overline{\dim}(F, u_0) \geqslant \dim(\ker L \cap \ker L^*),$$

and we are done by Theorem 2. □

5. Local Fredholm resolutions provide maximal constraints

Constraints, when they exist, may be far from easy to guess (see [8, Example 2.3.4]). But one can always write down a local nonlinear version of Fredholm scheme, which provides a full constraint if any such exists. Specifically, sticking to the above notations, let us recall known results [8]:

Definition 4. A *local Fredholm resolution* (or a *resolution*, for short) of F at 0 on $\mathcal{V}$ is a couple of maps (D, S) defined on $\mathcal{V}$ satisfying the identity:

$$\forall f \in \mathcal{V}, \quad F\big[S(f)\big] = f - D(f), \tag{6}$$

together with the following conditions:

(i) S (called the *approximate solution map*) is $\mathcal{E}$-valued with $S(0) = 0$;
(ii) D (called the *defect map*) is a submersion valued in a c_0-dimensional real vector subspace of $\mathcal{F}$, with $D(0) = 0$.

Furthermore, such a resolution is called *strong*,[3] provided $\ker L$ is complemented in $\mathcal{E}$ by a (closed) subspace $\mathcal{Z}$ such that: $D \circ F = 0$ on $\mathcal{U} \cap \mathcal{Z}$.

Given a resolution (D, S) for F at 0 on $\mathcal{V}$ and a k-constraint K for F on $\mathcal{V}$, the following inclusions obviously hold:

$$D^{-1}(0) \subset F(\mathcal{U}) \subset K^{-1}(0), \tag{7}$$

the countersets $D^{-1}(0)$ and $K^{-1}(0)$ are submanifolds of $\mathcal{V}$ passing through the origin, of codimension respectively c_0 and k, and 0 is a first order zero of $D \circ F$ [8, Proposition 2]. Finally, we have the following result:

Proposition 1. *(See [8, Theorems 2 and 6].) There exists a strong resolution for F at* 0*. Moreover, the defect map of any resolution is a full constraint whenever a full constraint exists.*

Here, we wish to investigate an extension of the second part of Proposition 1 in case there is no full constraint, that is when $\operatorname{cod}(F, 0) < c_0$. We will show that one can always project the defect map of a (strong) resolution onto a $\operatorname{cod}(F, 0)$-dimensional subspace of $\mathcal{F}$ and get a maximal constraint (up to a small correction under control). We require a definition.

Definition 5. A well-approximate constraint for F at 0 on $\mathcal{V}$, is a submersion $\mathcal{K}$ from a neighborhood $\mathcal{V}$ of 0 in $\mathcal{F}$ to a $\operatorname{cod}(F, 0)$-dimensional real vector space, with $\mathcal{K}(0) = 0$ and the two following properties:

(i) there exists a splitting $\mathcal{E} = \ker L \oplus \mathcal{Z}$ such that the restriction of $(\mathcal{K} \circ F)$ to $\mathcal{Z}$ vanishes near 0;
(ii) for each hypo-critical $\xi \in \ker L$ and for $t \in \mathbb{R}$ small, $(\mathcal{K} \circ F)(t\xi) = O(t^{2j+2})$ where $j = j_F(\xi)$.

Theorem 5. *Let (D, S) be a resolution for F at* 0 *on $\mathcal{V}$. Set $\mathbb{V}_0 \subset \mathcal{F}$ for the c_0-dimensional range of D. There exists a neighborhood $\mathcal{V}' \subset \mathcal{V}$ of* 0 *in $\mathcal{F}$, a linear projection Π of $\mathbb{V}_0$ onto a* $\operatorname{cod}(F, 0)$*-dimensional subspace $\mathbb{V}_1$ of $\mathbb{V}_0$, a map $\pi : \mathcal{V} \to D^{-1}(0) \cap \mathcal{V}$ reducing to the identity on $D^{-1}(0)$ (nonlinear projection), and a map N, defined near* 0 *in $D^{-1}(0) \times \ker \Pi$, valued in $\mathbb{V}_1$, such that the composed map*:

$$\Pi \circ D - N \circ \big[\pi, (1 - \Pi) \circ D\big] : \mathcal{V}' \to \mathbb{V}_1,$$

(*where* 1 *stands for the identity map*) *is a maximal constraint.*

Furthermore, if the resolution (D, S) is strong, then $\Pi \circ D : \mathcal{V}' \to \mathbb{V}_1$ is a well-approximate constraint.

Proof. Fixing bases, we may take D ranging in $\mathbb{R}^{c_0}$ and let $K : \mathcal{V} \to \mathbb{R}^{c_1}$ be a maximal constraint for F on $\mathcal{V}$, thus with $c_1 = \operatorname{cod}(F, 0)$. To spare notations in intermediate steps of the proof, we will freely keep the same letter $\mathcal{V}$ for

[3] A misprint occurs in [8, p. 26, Definition 3] where n_0-codimensional is meant for $\mathcal{Z}$.

neighborhoods of 0 in $\mathcal{F}$ shrunk as necessary; the final such $\mathcal{V}$ will be the neighborhood $\mathcal{V}' \subset \mathcal{V}$ whose existence is asserted by the theorem. Despite $\mathcal{F}$ being a Fréchet space instead of a Banach one, the local structure at 0 of the submersions D and K is classical [15, Chapter 1] because these maps have finite rank. So there exists maps,

$$\Psi_i : \mathcal{V} \to \mathcal{F}_i \oplus \mathbb{R}^{c_i}, \quad \text{with } \Psi_i(0) = 0 \text{ and } i \in \{0, 1\},$$

where $c_0 > c_1$ (if $c_1 = c_0$ we are done by Proposition 1) and each $\mathcal{F}_i$ is a closed factor of $\mathcal{F}$, with Ψ_i a diffeomorphism onto its image such that, setting p_1 (resp. p_2) for the canonical projections of $\mathcal{F}_i \oplus \mathbb{R}^{c_i}$ onto its first (resp. second) factor,[4] the following commutative diagrams relations hold:

$$D = p_2 \circ \Psi_0, \qquad K = p_2 \circ \Psi_1.$$

Define the nonlinear projection $\pi : \mathcal{V} \to D^{-1}(0) \cap \mathcal{V}$ by:

$$\pi = (\Psi_0)^{-1} \circ (p_1 \circ \Psi_0, 0).$$

Set $(f_0; D_1, \ldots, D_{c_0}) \mapsto (f_1; K_1, \ldots, K_{c_1})$ for the composed map $\Psi_1 \circ \Psi_0^{-1}$ considered between appropriate neighborhoods of the origins. From the inclusions (7) the K_α's vanish at $(f_0; 0)$, hence the rank of the matrix,

$$\left(\frac{\partial K_\alpha}{\partial D_j}(0; 0)\right), \quad \text{with } 1 \leqslant j \leqslant c_0,\ 1 \leqslant \alpha \leqslant c_1,$$

is equal to c_1. We may further arrange for (the $\mathbb{R}^{c_i}$ components of) the maps Ψ_i to be such that, setting:

$$D_N = (D_1, \ldots, D_{c_1}) =: (D_\beta)_{1 \leqslant \beta \leqslant c_1},$$
$$D_T = (D_{c_1+1}, \ldots, D_{c_0}) =: (D_b)_{c_1+1 \leqslant b \leqslant c_0},$$

the submatrix $(\frac{\partial K_\alpha}{\partial D_b})$ vanishes at the origin $= (0_f; 0_N, 0_T)$. In particular then, the submatrix $(\frac{\partial K_\alpha}{\partial D_\beta}(f_0; D_N, D_T))$ may be taken invertible on $\Psi_0(\mathcal{V})$. Now, the implicit function theorem provides locally functions $N_1, \ldots, N_{c_1}$ such that:

$$\forall \alpha \leqslant c_1, \quad K_\alpha(f_0; D_N, D_T) = 0 \quad \Longleftrightarrow \quad \forall \alpha \leqslant c_1, \quad D_\alpha = N_\alpha(f_0; D_T)$$

with the N_α's satisfying $N_\alpha(f_0; 0) = 0$ and the criticality condition:

$$\forall \alpha \leqslant c_1, \quad dN_\alpha(0; 0) = 0. \tag{8}$$

So the first part of Theorem 5, indeed, holds in the resulting neighborhood $\mathcal{V}'$, setting Π for the projection: $(D_N, D_T) \in \mathbb{R}^{c_0} \to D_N \in \mathbb{R}^{c_1}$ and N for the map (defined for D_T close enough to 0, valued in $\mathbb{R}^{c_1}$):

$$(f, D_T) \in \left(D^{-1}(0) \cap \mathcal{V}'\right) \times \ker \Pi \to N(f, D_T) = \left[N_\alpha\left(\Psi_0(f), D_T\right)\right]_{1 \leqslant \alpha \leqslant c_1}.$$

To prove the second part, we first note that, from the strongness assumption, $(D \circ F)$ vanishes in the direction of a subspace $\mathcal{Z}$ complementing $\ker L$. We will thus restrict $(D \circ F)$ to a hypo-critical direction defined by some vector $\xi \in \ker L$. For $t \in \mathbb{R}$ small, let us consider the composed function:

$$q(t) = N\left[\pi, (1 - \Pi) \circ D\right] \circ F(t\xi)$$

which satisfies $q(0) = q'(0) = 0$. Define an auxiliary function P by setting $q(t) = P \circ F(t\xi)$. We may apply Lemma 1 below with $H = P, G = F$ (thus with $j = j_F(\xi)$); from (8) we have $dP(0) = 0$ which implies $q(t) = O(t^{2j+2})$ as required. □

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I would like to thank Gilles Lebeau for an illuminating remark on [8, Theorem 3] prompting me toward Theorem 1.

[4] We avoid heavier notations like p_{1i}, p_{2i}.

Appendix A. Taylor expansion of a composed function at a critical point

The following lemma is used in the proofs of the Theorems 2 and 5 above:

Lemma 1. *Let $\mathcal{E},\mathcal{G},\mathcal{H}$ be Fréchet spaces, G (resp. H) be a map defined near 0 in $\mathcal{E}$ (resp. $\mathcal{G}$) valued in $\mathcal{G}$ (resp. $\mathcal{H}$), with $G(0)=0$, $H(0)=0$. For each $\xi\in\mathcal{E}$ and for $t\in\mathbb{R}$ small enough, set formally: $G(t\xi)=\sum_{k=1}^{\infty}\frac{t^k}{k!}g_k(\xi)$ and let $j=j_G(\xi)$ be the first nonzero integer (if finite) such that: $\forall k\leqslant j,\ g_k(\xi)=0$ and $g_{k+1}(\xi)\neq 0$. Then the following expansion holds:*

$$(H\circ G)(t\xi)=\sum_{k=j+1}^{2j+1}\frac{t^k}{k!}dH(0)\big[g_k(\xi)\big]+O\big(t^{2j+2}\big).$$

Proof. If $j=0$, the lemma is trivial, so we assume $j\geqslant 1$ (in other words, ξ critical for G) and, of course, j finite. For each integer $k\geqslant 2$, Faa di Bruno's formula applied repeatedly shows that $\frac{d^k}{dt^k}[(H\circ G)(t\xi)]_{t=0}$ is a linear combination (with universal rational coefficients) of terms typically like: $d^rH(0)[g_{s_1}(\xi),\dots,g_{s_r}(\xi)]$, with: $r\geqslant 1$, $s_1+\cdots+s_r=k$ and: $\forall i\in\{1,\dots,r\}$, $s_i\geqslant j+1$ (from the definition of $j=j_G(\xi)$). The latter implies $k\geqslant r(j+1)$; in particular, $k\geqslant 2j+2$ if $r>1$, which proves the lemma. □

Appendix B. On the underdetermined elliptic case

It is interesting to compare the results of the present paper with those, less complete, known on the local solvability of underdetermined elliptic operators. We thus provide a parallel account on the latter. Let $V(\mathcal{E})$ and $V(\mathcal{F})$ denote the vector bundles over M corresponding to the Fréchet spaces of sections $\mathcal{E}$ and $\mathcal{F}$ respectively. So far, $V(\mathcal{E})$ and $V(\mathcal{F})$ had equal rank; we now consider the case rank $V(\mathcal{E})>$ rank $V(\mathcal{F})$. When so, a linear differential operator $A:\mathcal{E}\to\mathcal{F}$ is called underdetermined elliptic if, for each $m\in M$ and nonzero $\xi\in T_m^*M$, the principal symbol of A at (m,ξ), a linear map $\sigma_A(m,\xi):V(\mathcal{E})_m\to V(\mathcal{F})_m$, is *onto*.[5] For completeness, let us recall that $\sigma_A(m,\xi)$ may be defined as follows: given $U\in V(\mathcal{E})_m$, pick $u\in\mathcal{E}$ such that $u(m)=U$ and a function $\theta:M\to\mathbb{R}$ such that $\theta(m)=0$, $d\theta(m)=\xi$; then

$$\sigma_A(m,\xi)(U)=A\left(\frac{\theta^r}{r!}u\right)(m)\in V(\mathcal{F})_m,$$

where $r\in\mathbb{N}^*$ stands for the order of the operator A. One readily verifies that the preceding right-hand side depends only on ξ and (linearly) on U.

Fixing auxiliary L^2 scalar products on $\mathcal{E}$ and $\mathcal{F}$ (both denoted by $\langle .,.\rangle$) and setting A^* for the formal adjoint of A, we have:

Fact 1. $\ker A^*=\ker AA^*$; *in particular,* $\dim(\ker A^*)$ *must be finite.*

Indeed, fix $f\in\ker AA^*$; on the one hand, $A^*f\in\ker A$, on the other hand:

$$\forall\zeta\in\ker A,\quad \langle\zeta,A^*f\rangle=\langle A\zeta,f\rangle=0, \tag{9}$$

in other words, $A^*f\in(\ker A)^{\perp}$; altogether $A^*f=0$ hence, since f is arbitrary, $\ker AA^*\subset\ker A^*$. The reversed inclusion is trivial, so the asserted equality holds. Finally, we observe that the operator AA^* is *elliptic*; as such, it has a finite-dimensional kernel [3, Appendix].

Fact 2. $\operatorname{Im}A=(\ker A^*)^{\perp}$; *in particular,* $\operatorname{Im}A$ *is closed and complemented in* $\mathcal{F}$. *Moreover,* $\operatorname{Im}A^*$ *is closed, complemented in* $\mathcal{E}$, *equal to* $(\ker A)^{\perp}$.

[5] Whereas it is an isomorphism in the (determined) elliptic case.

Indeed, observing that the operator AA^* is formally self-adjoint and setting P_* for the orthogonal projection of $\mathcal{F}$ onto $\ker A^*$, we note that Fredholm theorem [3, Appendix] combined with Fact 1 implies:

$$\forall f \in \mathcal{F}, \quad \exists u \in \mathcal{E}, \quad AA^*u = f - P_* f.$$

In particular, we infer $(\ker A^*)^\perp \subset \operatorname{Im} A$; since the reversed inclusion holds (arguing as in (9)), we get the desired equality. The next assertion classically follows [17, Chapter 5] and so does the final one [13, p. 234]. In particular, we get that an underdetermined elliptic operator is semi-Fredholm [13, p. 230].

Back to our current notations, thus with $F : \mathcal{U} \subset \mathcal{E} \to \mathcal{F}$, we now assume that $L = dF(0)$ is underdetermined elliptic, with nonzero cokernel (finite-dimensional, by Fact 1). For u close to 0 in $\mathcal{E}$, the linearized operator $dF(u)$ stays underdetermined elliptic: we set $c_u(F) = \dim(\ker[dF(u)]^*)$. We stick to Definition 1 (k-constraint, thus with $k \leqslant c_0$ still, by Fact 2 for $A = L$), here with the abbreviation $c_0 := c_0(F) = \dim(\ker L^*) \neq 0$, and to Definition 4 (LFR) except for its last part (strongness) since we do not have information on $\ker L$ any longer.

Fact 3. *There exists a local Fredholm resolution for F at 0. Moreover, the defect map is a full constraint provided there exists one.*

Indeed, if (D, S) is an LFR for the elliptic operator $f \mapsto G(f) := F(L^* f)$ defined near 0 in $\mathcal{F}$ (such an LFR exists by Proposition 1), then $(D, L^* \circ S)$ is an LFR for F at 0. It is such that, for each k-constraint K on F, the local inclusion $D^{-1}(0) \subset K^{-1}(0)$ holds (obvious); if $k = c_0$, arguing as in [8, pp. 26–27] with the help of Fact 2 (for $A = L$), we infer $D^{-1}(0) = K^{-1}(0)$, which completes the proof of Fact 3.

Fact 4. *The operator $G = F \circ L^*$ admits a full constraint near 0 in $\mathcal{F}$.*

Indeed, by construction $dG(0)$ is formally self-adjoint, hence $dG(f)$ has zero index for f close to 0 [13, p. 285] and, recalling Fact 1, $\dim\ker dG(0) = c_0(F)$. Near 0 in $\mathcal{F}$, we thus have [13, p. 235] (with an obvious notation): $c_f(G) \leqslant c_0(F)$ while, still by construction, $\ker L^* \subset \ker dG(f)$. Therefore $\ker dG(f) \equiv \ker L^*$ and Fact 4 follows from [8, Theorem 3].

From Fact 4 we see that, G cannot be used (in the manner of Section 3) to discuss the loss of constrainability of F.

Fact 5. *The restriction of F to $(\ker L)^\perp$ admits a full constraint near 0 in $\mathcal{F}$.*

Indeed, if $u \in (\ker L)^\perp$, we can solve uniquely for $f \in (\ker L^*)^\perp$ the equation $L^* f = u$, by solving instead: $LL^* f = Lu$. The latter equation is elliptic hence admits a solution, by Fredholm theorem [3, Appendix], due to Fact 2 (for $A = L$). In particular, we get $(L^* f - u) \in \ker L$. But $\operatorname{Im} L^* \subset (\ker L)^\perp$ as in (9), so $(L^* f - u) \in (\ker L)^\perp$ and $(L^* f - u) = 0$ as desired. We just proved the existence of a map $S^* : (\ker L)^\perp \to (\ker L^*)^\perp$ such that $F|_{(\ker L)^\perp} = G \circ S^*$. Now Fact 5 follows from Fact 4.

Fact 6. *If F admits a full constraint near 0 in $\mathcal{F}$, then $c_u(F) \equiv c_0$.*

Indeed, we may argue as in Section 2 (the "only if" part of Theorem 1) because semi-Fredholm operators share with Fredholm ones the same stability properties [13, p. 235]. We conjecture that the converse of Fact 6 holds.

Finally, let us provide a couple of examples. Fix a Riemannian metric g on the (compact connected) manifold M and set $|.|, \omega, \delta, \Delta$, respectively for the following quantities associated to g: norm on tensors, volume form, co-differential, Laplacian $\Delta = d\delta + \delta d$. Take $\mathcal{E} = \Omega^1(M)$ (the 1-forms) and $\mathcal{F} = \Omega^0(M)$ (the real functions), writing Ω^0 and Ω^1 for short; denote by $\Lambda_k \subset \Omega^0$ the kth eigenspace of Δ (so Λ_0 stands for the constant functions on M).

Example 1. Near 0 in Ω^1, consider the differential operator $F_1 : \Omega^1 \to \Omega^0$ given by $F_1(\alpha) = \delta(e^{-|\alpha|^2}\alpha)$. An easy calculation yields $L = dF_1(0) = \delta$ and the principal symbol: $\sigma_L(m, \xi) : \alpha \in T_m^* M \to g_m(\alpha, \xi) \in \underline{\mathbb{R}}_m$ is readily onto; so F_1 is underdetermined elliptic at 0. Moreover, using the L^2 scalar products defined by the metric g, we have $L^* = d : \Omega^0 \to \Omega^1$ and $LL^* = \Delta : \Omega^0 \to \Omega^0$ which is elliptic. From $L^* = d$ we get $\ker L^* = \Lambda_0$ and $c_0 = 1$. Since

F_1 satisfies: $\forall\alpha \in \Omega^1$, $\int_M F_1(\alpha)\omega = 0$, we infer that F_1 is fully constrainable near 0 in Ω^0. Last, a straightforward calculation yields:

$$\big[dF_1(\alpha)\big]^*(v) = e^{-|\alpha|^2}\big[dv - 2g(\alpha, dv)\alpha\big],$$

showing that $\ker[dF_1(\alpha)]^* \equiv \Lambda_0$ and $c_\alpha(F_1) \equiv 1$.

Example 2. Consider now the operator: $\alpha \in \Omega^1 \to F_2(\alpha) = e^{-|\alpha|^2}\delta\alpha \in \Omega^0$ near $\alpha = 0$. We still have $L = dF_2(0) = \delta$, hence $c_0 = 1$.

Theorem 6. *Assume the existence of $\beta \in \Omega^1$ such that the set*

$$\big\{m \in M,\ d\big[(\delta\beta)\beta\big](m) = 0\big\}$$

has zero measure. Then F_2 admits no constraint near 0 in Ω^0.

Proof. By Fact 6 it is enough to prove that $c_\alpha(F_2) \not\equiv 1$ for α near 0 in Ω^1. Routine calculation yields:

$$\big[dF_2(\alpha)\big]^*(v) = e^{-|\alpha|^2}\big[dv - vd\big(|\alpha|^2\big) - 2v(\delta\alpha)\alpha\big].$$

Pick $\alpha = \gamma := t\beta$ with β fulfilling the assumption and $t \in \mathbb{R}^*$ small. Suppose that $\ker[dF_2(\gamma)]^*$ contains a nonzero function w. On the set $\{w \neq 0\}$, which has positive measure, we get the equation:

$$\frac{dw}{w} - d\big(|\gamma|^2\big) = 2(\delta\gamma)\gamma,$$

hence also $d[(\delta\gamma)\gamma] = 0$, contradicting the assumption. Therefore $c_\gamma(F_2) = 0$ and we are done. □

Exercise. The assumption of Theorem 6 is fulfilled in dimension 2 by the 1-form $\beta = d\varphi + \delta(\varphi\omega)$ with $\varphi \in \Omega^0$ such that the set $\{\Delta(\varphi^2) = 0\}$ has zero measure (for instance with $\varphi \in \Lambda_1$ on the standard 2-sphere).

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